The generalized Davey-Stewartson equations, its Kac-Moody-Virasoro symmetry algebra and relation to DS equations

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Abstract

We compute the Lie symmetry algebra of the generalized Davey-Stewartson (GDS) equations and show that under certain conditions imposed on parameters in the system it is infinite-dimensional and isomorphic to that of the standard integrable Davey-Stewartson equations which is known to have a very specific Kac-Moody-Virasoro loop algebra structure. We discuss how the Virasoro part of this symmetry algebra can be used to construct new solutions, which are of vital importance in demonstrating existence of blow-up profiles, from known ones using Lie subgroup of transformations generated by three-dimensional subalgebras, namely $\mathrm{sl}(2,\mathbb{R})$. We further discuss integrability aspects of GDS equations.

1 Introduction

A system of nonlinear partial differential equations in 2+1 dimensions as a model of wave propagation in a bulk medium composed of an elastic material with couple stresses has recently been derived in Ref. [1], namely

$$\begin{aligned} & i\psi_t + \delta\psi_{xx} + \psi_{yy} = \chi|\psi|^2\psi + \gamma(w_x + \phi_y)\psi \\ & w_{xx} + n\phi_{xy} + m_2 w_{yy} = (|\psi|^2)_x \\ & n w_{xy} + \lambda\phi_{xx} + m_1\phi_{yy} = (|\psi|^2)_y, \end{aligned} \tag{1.1}$$

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with the condition $(\lambda - 1)(m_1 - m_2) = n^2$. Here $\psi(t, x, y)$ is a complex function, $w(t, x, y)$ and $\phi(t, x, y)$ are real functions and $\delta, n, m_1, m_2, \lambda, \chi, \gamma$ are real constants. The authors of [1] showed that if the parameters are related by

$$n = 1 - \lambda = m_1 - m_2, \tag{1.2}$$

then (1.1) can be reduced to the standard Davey-Stewartson (DS) equations (in general not integrable) by a non-invertible point transformation of dependent variables. Therefore, they called (1.1) the generalized Davey-Stewartson (GDS) equations. Below, we justify this naming from a group-theoretical point of view. Also, in [1] some travelling type solutions of (1.1) in terms of elementary and elliptic functions are obtained. Based on some physically obvious Noetherian symmetries (time-space translations and constant change of phase), global existence and nonexistence results are given in [2]. In another recent work [3], under some constraints on the physical parameters, the so-called hyperbolic-elliptic-elliptic case of the system (1.1) (In [3] the system is classified into different types according to the signs of parameters $(\delta, m_1, m_2, \lambda)$) was shown to admit singular solutions that blow up in a finite time. To do this, inspired by the (pseudo) conformal invariance of DS system, they used the fact that time-dependent $\mathrm{SL}(2, \mathbb{R})$ invariant solutions can be generated from stationary radial solutions for an appropriate choice of coefficients.

The purpose of this article is to study GDS equations from group theoretical point of view. For our purposes, we find it more convenient to consider the differentiated form of (1.1). Thus, differentiating the last two equations of (1.1) with respect to x and y, respectively and then making the substitution $w_x \to w$, $\phi_y \to \phi$ and rewriting the corresponding system in a real form by separating $\psi = u + iv$ into real and imaginary parts, we obtain a system of four real partial differential equations

$$\begin{aligned} u_t + \delta v_{xx} + v_{yy} &= \chi v(u^2 + v^2) + \gamma v(w + \phi), \\ -v_t + \delta u_{xx} + u_{yy} &= \chi u(u^2 + v^2) + \gamma u(w + \phi), \\ w_{xx} + n\phi_{xx} + m_2 w_{yy} &= 2(u_x^2 + u u_{xx} + v_x^2 + v v_{xx}), \\ n w_{yy} + \lambda \phi_{xx} + m_1 \phi_{yy} &= 2(u_y^2 + u u_{yy} + v_y^2 + v v_{yy}). \end{aligned} \tag{1.3}$$

In the sequel, we shall call (1.3) the GDS equations.

The main result of the paper is to show that, when some conditions on physical parameters $\delta, n, m_1, m_2, \lambda$ are imposed, the Lie algebra of the symmetry group of the GDS system has a Kac-Moody-Virasoro (KMV) loop structure which is shared by the symmetry algebras of all known integrable equations in 2+1 dimensions such as the Kadomtsev-Petviashvilli (KP) equation [4, 5] and the usual integrable DS equations [6]. The corresponding special case is candidate for being integrable. Moreover, we

show that this algebra is isomorphic to that of DS equations [6]

$$\begin{aligned} &i\psi_t + \delta_1\psi_{xx} + \psi_{yy} = \delta_2|\psi|^2\psi + w\psi \\ &\varepsilon_1 w_{xx} + w_{yy} = \varepsilon_2(|\psi|^2)_{yy}, \end{aligned} \tag{1.4}$$

with $\delta_1 = \pm 1$, $\delta_2 = \pm 1$. In the special case when $\delta_1 + \varepsilon_1 = 0$, this system which is one of the rare systems in more than 1+1 dimensions for which the Cauchy initial value problem is solvable by the inverse spectral transform (IST) technique becomes completely integrable. The Lie algebra of the symmetry group of the integrable DS system is referred to as the DS algebra. This isomorphism (a necessary condition for two different systems to be transformable into each other) should motivate one to look for point transformations taking the Lie algebras into each other. We expect such transformations to transform the systems into each other as well.

In Section 2 we compute the Lie symmetry algebra of (1.3) and identify its structure. In particular, we show that for special choice of parameters it is a centerless Kac-Moody-Virasoro algebra.

2 The symmetry group of the GDS equations and structure of its Lie algebra

We apply the standard infinitesimal procedure [7] to find the symmetry algebra L and hence the symmetry group G of (1.3). We write the GDS equations as a system $\Delta_i(t, x, y, u, v, w, \phi) = 0$, $i = 1, 2, 3, 4$. A general element of the algebra is represented by a vector field

$$\mathbf{V} = \tau\partial_t + \xi\partial_x + \eta\partial_y + \varphi_1\partial_u + \varphi_2\partial_v + \varphi_3\partial_w + \varphi_4\partial_\phi, \tag{2.1}$$

where the coefficients $\tau, \xi, \eta, \varphi_i, i = 1, 2, 3, 4$ are functions of t, x, y, u, v, w, ϕ. According to the general theory for symmetries of differential equations, to find these functions we prolong the vector field (2.1) to second order derivatives and require that the second prolonged vector field annihilates Δ_i on the solution manifold of the system, namely

$$\mathrm{pr}^{(2)}\mathbf{V}(\Delta_i(t, x, y, u, v, w, \phi))\Big|_{\Delta_i=0} = 0, \quad i = 1, 2, 3, 4, \tag{2.2}$$

where $\mathrm{pr}^{(2)}\mathbf{V}$ is the second prolongation of the vector field $\mathbf{V}$. This condition provides us with a quite complicated system of determining equations (a system of linear partial differential equations) for the coefficients. This step is entirely algorithmic and is implemented on several computer algebra packages like REDUCE, MATHEMATICA,

MAPLE (See [8] for a survey of symbolic softwares for symmetry). The final step of integrating the determining equations is less algorithmic. Solving these huge number of determining equations we find that the general element can be written as

$$\mathbf{V} = T(f) + X(g) + Y(h) + W(m), \tag{2.3}$$

where

$$\begin{aligned} T(f) &= f(t)\partial_t + \frac{1}{2}f'(t)(x\partial_x + y\partial_y - u\partial_u - v\partial_v - 2w\partial_w - 2\phi\partial_\phi) \\ &\quad - \frac{(x^2+\delta y^2)}{8\delta}[f''(t)(v\partial_u - u\partial_v) + \frac{f'''(t)}{2\gamma}(\partial_w + \partial_\phi)], \\ X(g) &= g(t)\partial_x - \frac{x}{2\delta}[g'(t)(v\partial_u - u\partial_v) + \frac{g''(t)}{2\gamma}(\partial_w + \partial_\phi)], \\ Y(h) &= h(t)\partial_y - \frac{y}{2}[h'(t)(v\partial_u - u\partial_v) + \frac{h''(t)}{2\gamma}(\partial_w + \partial_\phi)], \\ W(m) &= m(t)(v\partial_u - u\partial_v) + \frac{m'(t)}{2\gamma}(\partial_w + \partial_\phi). \end{aligned} \tag{2.4}$$

The functions $g(t)$, $h(t)$, and $m(t)$ are arbitrary functions of class $C^\infty(I)$, $I \subseteq \mathbb{R}$. The function $f(t)$ is arbitrary if

$$m_2\delta + n + 1 = 0, \quad m_1\delta + n\delta + \lambda = 0, \tag{2.5}$$

otherwise $f(t) = c_2t^2 + c_1t + c_0$. We mention that these conditions come from the fact that two of the determining equations are

$$(m_2\delta + n + 1)f'''(t) = 0, \quad (m_1\delta + n\delta + \lambda)f'''(t) = 0,$$

whereas the remaining ones are solved without any constraints on g, h and m.

We mainly focus on the case when $f(t)$ is allowed to be arbitrary. The symmetry algebra realized by the vector fields (2.3) and (2.4) is then infinite-dimensional and more important has the structure of a Kac-Moody-Virasoro algebra as we shall see below. More interestingly, it is generic among the symmetry algebras of a few 2+1-dimensional integrable partial differential equations (the KP equation, the modified KP equation, the potential KP equation, the integrable three-wave resonant equations and the integrable DS equations). Henceforth, we shall call this the GDS symmetry algebra and the corresponding system the GDS system.

Note that sometimes it is more convenient to use the polar decomposition $u+iv = Re^{i\sigma}$ so that in (2.4) we can write

$$u\partial_u + v\partial_v = R\partial_R, \quad -(v\partial_u - u\partial_v) = \partial_\sigma.$$

The commutation relations for the GDS algebra are easily obtained as follows:

$$\begin{aligned}
&[T(f_1), T(f_2)] = T(f_1 f_2' - f_1' f_2) \\
&[T(f), X(g)] = X(f g' - \frac{1}{2} f' g) \\
&[T(f), Y(h)] = Y(f h' - \frac{1}{2} f' h) \\
&[T(f), W(m)] = W(f m') \\
&[X(g_1), X(g_2)] = -\frac{1}{2\delta} W(g_1 g_2' - g_1' g_2) \\
&[Y(h_1), Y(h_2)] = -\frac{1}{2} W(h_1 h_2' - h_1' h_2) \\
&[X(g), Y(h)] = [X(g), W(m)] = [Y(h), W(m)] = [W(m_1), W(m_2)] = 0.
\end{aligned} \tag{2.6}$$

From (2.6) we see that the GDS system has a Lie symmetry algebra L isomorphic to that of the DS symmetry algebra [6]. Indeed, it allows a Levi decomposition

$$L = S \ni\!\!\!\!+ N, \tag{2.7}$$

where $S = \{T(f)\}$ is a simple infinite dimensional Lie algebra and

$$N = \{X(g), Y(h), W(m)\}$$

is a nilpotent ideal (nilradical). Here, $\ni\!\!\!\!+$ denotes a semi-direct sum. The algebra $\{T(f)\}$ is isomorphic to the Lie algebra corresponding to the Lie group of diffeomorphisms of a real line.

We remark that a similar isomorphism between the symmetry algebras of a class of (integrable) generalized cylindrical KP (GCKP) equation and of the KP equation was pointed out in Ref. [9].

Expanding the arbitrary functions f, g, h and m into Laurent polynomials and considering each monomial t^n (n not necessarily positive integer) separately, we obtain a realization of a KMV algebra without central extension. Here the factor subalgebra S is the Virasoro part, the nilpotent subalgebra N is the Kac-Moody part of the GDS algebra [10]. We refer for different realizations of the Virasoro algebras to [11]. Furthermore, just as the DS algebra [6] it can be shown that the GDS algebra with (2.5) can be imbedded into a Kac-Moody-type loop algebra.

Theorem 2.1 *The system* (1.3) *is invariant under an infinite-dimensional Lie point symmetry group, the Lie algebra of which has a Kac-Moody-Virasoro structure isomorphic to the DS algebra if and only if the conditions* (2.5) *hold.*

Let us mention that the GDS equations are also invariant under a group of discrete transformations generated by

$$\begin{array}{llllll} t\to t, & x\to -x, & y\to y, & \psi\to\psi, & w\to w, & \phi\to\phi \\ t\to t, & x\to x, & y\to -y, & \psi\to\psi, & w\to w, & \phi\to\phi \\ t\to t, & x\to x, & y\to y, & \psi\to -\psi, & w\to w, & \phi\to\phi \\ t\to -t, & x\to x, & y\to y, & \psi\to\psi^*, & w\to w, & \phi\to\phi. \end{array} \tag{2.8}$$

The obvious physical symmetries L_p of the GDS equations are obtained by restricting all the functions f, g, h and m to be first order polynomials. Indeed, we have

$$\begin{aligned} &T = T(1) = \partial_t, \quad P_1 = X(1) = \partial_x, \quad P_2 = Y(1) = \partial_y \\ &W_0 = W(1) = v\partial_u - u\partial_v, \\ &D = T(t) = t\partial_t + \frac{1}{2}(x\partial_x + y\partial_y - u\partial_u - v\partial_v - 2w\partial_w - 2\phi\partial_\phi) \\ &B_1 = X(t) = t\partial_x - \frac{x}{2\delta}(v\partial_u - u\partial_v), \quad B_2 = Y(t) = t\partial_y - \frac{y}{2}(v\partial_u - u\partial_v) \\ &W_1 = W(t) = t(v\partial_u - u\partial_v) + \frac{1}{2\gamma}(\partial_w + \partial_\phi). \end{aligned} \tag{2.9}$$

We see that T, P_1, P_2 generate translations, D dilations, B_1 and B_2 Galilei boosts in the x and y directions, respectively. Finally, W_0 and W_1 generate a constant change of phase of ψ and a change of phase of ψ, linear in t, plus constant shifts in w and ϕ, respectively.

The generators (2.9) form a basis of a eight-dimensional solvable Lie algebra $L_p = \{D, T, P_1, P_2, B_1, B_2, W_0, W_1\}$. It has a seven-dimensional nilpotent ideal (the nilradical) $N = \{T, P_1, P_2, B_1, B_2, W_0, W_1\}$.

Another finite-dimensional algebra, not contained in L_p is obtained by restricting $f(t)$ to quadratic polynomials. We obtain $T = T(1)$, $D = T(t)$ as in (2.9), and in addition

$$C = T(t^2) = t^2\partial_t + tD - \frac{(x^2 + \delta y^2)}{4\delta}(v\partial_u - u\partial_v). \tag{2.10}$$

The commutation relations are

$$[T, D] = T, \quad [T, C] = 2D, \quad [D, C] = C,$$

so that we have obtained the algebra $\mathrm{sl}(2, \mathbb{R})$ with C generating conformal type of

transformations

$$\begin{aligned}
&\tilde{t} = \frac{t}{1-pt}, \quad \tilde{x} = \frac{x}{1-pt}, \quad \tilde{y} = \frac{y}{1-pt}, \\
&\tilde{R} = (1-pt)R, \quad \tilde{\sigma} = \frac{p(x^2+\delta y^2)}{4\delta(1-pt)} + \sigma, \\
&\tilde{w} = (1-pt)^2 w, \quad \tilde{\phi} = (1-pt)^2\phi,
\end{aligned} \tag{2.11}$$

where p is the group parameter. Further, composing (2.11) with time translations generated by T and dilations generated by D we obtain the $\mathrm{SL}(2,\mathbb{R})$ group generated by actions on the space of independent and dependent variables. It should be mentioned that any finite dimensional subalgebra of the Virasoro algebra of 2+1 dimensional integrable equations is isomorphic to $\mathrm{sl}(2,\mathbb{R})$ or one of its subalgebras. The transformed variables and the new solution in terms of the original ones are given by the formulas

$$\begin{aligned}
&\tilde{t} = \frac{c+dt}{a+bt}, \quad \tilde{x} = \frac{x}{a+bt}, \quad \tilde{y} = \frac{y}{a+bt}, \quad ad-bc=1 \\
&\tilde{\psi} = (a+bt)^{-1}\exp\{\frac{ib(x^2+\delta y^2)}{4\delta(a+bt)}\}\psi(\tilde{t},\tilde{x},\tilde{y}) \\
&\tilde{w} = (a+bt)^{-2} w(\tilde{t},\tilde{x},\tilde{y}) \\
&\tilde{\phi} = (a+bt)^{-2}\phi(\tilde{t},\tilde{x},\tilde{y}).
\end{aligned} \tag{2.12}$$

Here a, b, c are the group parameters of $\mathrm{SL}(2,\mathbb{R})$. These are exactly the formulas which played an essential role in the construction of analytic blow-up profiles [3] in which the authors made use of stationary radial solutions (ψ, w, ϕ) to generate new solutions (time dependent) $(\tilde{\psi}, \tilde{w}, \tilde{\phi})$ of the GDS equations. More generally, the elements of the connected part of the full symmetry group of the GDS equations can be obtained by integrating the vector fields (2.3), (2.4). We refer the reader to Ref. [6] for the general Lie group of transformations of DS algebra.

Let us now return to the isomorphic GDS and DS symmetry algebras, and transform the GDS vector fields (2.1) by the point transformation $q = w + \phi - |\psi|^2$. It is easy to see that the component $\frac{1}{2}(\partial_w + \partial_\phi)$ transforms to ∂_q, and $D \to x\partial_x + y\partial_y - u\partial_u - v\partial_v - 2q\partial_q$, and the rest remains unaltered, namely the DS symmetry algebra is obtained. This means that the functions (ψ, q) satisfy the DS equations whenever (ψ, w, ϕ) satisfy the GDS equations, but not vice versa. At this time, it remains open whether it is possible to construct an invertible point transformation relating these two systems.

We conclude by making several comments. As is well illustrated by the results of this paper, knowing that a nonlinear partial differential equation (or system) admits

a KMV algebra as a symmetry algebra can serve as a useful criterion of identifying integrable equations. In particular, this fact can be used to pick out an integrable equation from a class of generically nonintegrable ones. For instance, for all values of parameters not satisfying (2.5), the Virasoro part $T(f)$ of the GDS algebra is not present. The first author of the present paper and Winternitz [12] used the same approach to identify all subclasses invariant under a KMV algebra and its subalgebras containing up to three arbitrary functions of time from a rather general class of KP type equations involving 9 arbitrary functions of one or two variables. On the other hand, a classification of all one- and two-dimensional subalgebras of the DS algebra into conjugacy classes under the adjoint action of the DS group (including the discrete transformations) is performed in [6]. The GDS algebra will have the same conjugacy classes of subalgebras as the DS algebra. Depending on which of the functions $g(t)$, $h(t)$ and $m(t)$ are nonzero, precisely six conjugacy classes of one-dimensional subalgebras exist:

$$L_{1,1} = \{T(1)\}, \quad L_{1,2}^{a} = \{X(1) + aY(1)\}, \quad L_{1,3}(h) = \{X(1) + Y(h)\}, \quad a \geq 0,$$
$$L_{1,4} = \{Y(1)\}, \quad L_{1,5} = \{W(t)\}, \quad L_{1,6} = \{W(1)\}.$$

They can be used to reduce the integrable GDS system to integrable one in two variables and thus to obtain subgroup invariant solutions. There will be four type of reductions since only the first four subgroups corresponding to $(L_{1,1}, L_{1,2}^{a}, L_{1,3}(h), L_{1,4})$ will generate actions on the coordinate space (t, x, y). The remaining two $(L_{1,5}, L_{1,6})$ generate purely vertical (or gauge) transformations changing phases only and thus lead to no reductions. For example, one can show that all the travelling wave solutions obtained in [1] can be extracted from those of representative reduced equations by applying appropriate symmetry group transformations. We note that these type of physically important solutions are invariant under translational subgroups alone.

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